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Penrose limits and non-local theories

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Abstract: We investigate Penrose limits of two classes of non-local theories, little string theories and non-commutative gauge theories. Penrose limits of the near-horizon geometry of NS5-branes help to shed some light on the high energy spectrum of little string theories. We attempt to understand renormalization group flow in these theories by considering Penrose limits wherein the null geodesic also has a radial component. In particular, we demonstrate that it is possible to construct a plane-wave spacetime which interpolates between the linear dilaton and the AdS regions for the type-IIA NS5-brane. Similar analysis is considered for the holographic dual geometry to non-commutative field theories.

Keywords: Renormalization Group, M-Theory, Penrose limit and pp-wave background.

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1. Introduction

String theory has provided natural candidates for non-gravitational theories which are nonetheless non-local. Prominent examples in this category of theories are non-commutative field/string theories [1]–[5] and little string theories [6, 7]. Understanding the details of dynamical behaviour in these theories might shed light on the consequences of non-local interactions in a quantum theory, thereby hopefully paving the way toward quantum gravity.

A very useful tool to understand the dynamics of these non-local theories is to use their holographic description, wherein one can talk about string propagation in a background spacetime. In the case of little string theories one is led to considering string theories in linear dilaton backgrounds [8]. For non-commutative theories, the holographic dual is the near horizon geometry of the D-brane configuration realizing the gauge theory in the low energy limit. In the case of four-dimensional non-commutative theories, this is a specific deformation of the AdS_5 background [9, 10]. The holographic description has been useful

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to understand some of the characteristic features of these theories. For instance, one can compute correlation functions of operators in the supergravity limit, as in [8, 11, 12] for little string theories and as in [10] for non-commutative theories.

While the holographic description has many positive features, a full solution of string theory in the backgrounds in question is lacking. So it is fair to say that one does not have a complete picture of the dual theory. However, it might be possible that if one isolated a certain sector of the field theory, one could hope for a solution within that particular sector. To this end, one needs to identify a solvable sector of the field theory, which *a priori* is not an easy task. Instead, one can draw inspiration from the gravity dual.

In general relativity, just as one can consider zooming in onto a point in a manifold, thus giving rise to the important concept of tangent space, one can consider zooming in onto a null geodesic in a spacetime. While the former is flat, the latter has more structure, essentially coming from the direction along the geodesic; in particular, it gives rise to a curved space known as a plane-parallel wave, or plane-wave, as was demonstrated by Penrose [13] several decades ago.

More specifically, in taking the so-called Penrose limit of a spacetime, one boosts to the speed of light, which would typically tend to make the curvature tensor components diverge, but at the same time, one also scales up the metric by "zooming in" infinitesimally close to the geodesic, which scales the curvature back down. One may also regard this as the limiting procedure in which one considers timelike observers going faster and faster, but with their clocks appropriately recalibrated. By scaling up a coordinate transverse to the direction along the geodesic, one obtains a Killing field; in fact, one can show that this will be a covariantly constant null Killing field. This is the essential ingredient to the plane-wave geometry.

Recently, a very interesting solvable model of string theory in Ramond-Ramond backgrounds was proposed in [14, 15]. This was achieved by taking the Penrose limit [13] of $AdS_5 \times \mathbf{S}^5$ spacetime [15, 16], the holographic dual of $d = 4$, $\mathcal{N} = 4$ Super-Yang-Mills theory. In this limit one finds that one is concentrating on a particular sector of the gauge theory, with large dimensions of operators and large charges, but with a finite difference between the charges and the dimensions. Since the dual background is exactly solvable as a string theory, we can claim to have understood, at least in principle, this particular sector of the gauge theory.

Since the new scaling limit of $\mathcal{N} = 4$ Yang-Mills defined in [15] yielded new insights into the gauge/string correspondence, we can ask whether a similar strategy will lead to something useful in the context of little strings. In fact, the answer is affirmative and we will shortly see how we can obtain some information about the high energy spectrum of little strings. This will be achieved by considering the Penrose limit of the linear dilaton geometry, such that on the little string side we are concentrating on the sector of the theory with energies of order $\sqrt{N}l_s$ and carrying a U(1) charge of order N. N is the usual dimensionless number characterizing the little string theory, which we shall take large so as to trust the supergravity approximation. For this sector of the little string theory, we will find that the spectrum is just the free string spectrum in 10 dimensions.

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We then turn to an investigation of renormalization group trajectories in the non-local theories in question. Under the holographic map, the renormalization group flow in the field theory side is associated with the radial evolution in the spacetime geometry. Thus, if we were to consider null geodesics which probe the radial direction, we could gain some intuition about the RG flows in the dual field theory. We will construct Penrose limits of null geodesics with radial component in the cigar geometry to study the RG flow in little string theory with an infra-red cut-off. In addition, we will show that in the plane-wave limit, it is possible to construct an RG trajectory that interpolates between the linear dilaton region and the AdS_7 region corresponding to the infra-red $(2,0)$ superconformal field theory, for the case of the type-IIA little string theory. We will see that the resulting plane-wave geometries are time-dependent, this time dependence being interpreted as the scale dependence of the dual theory. A similar situation can be studied for non-commutative theories and we also comment on constructing such geometries.

The organization of the paper is as follows. We begin in section 2 with a general introduction to Penrose limits, concentrating on a particular class of metrics which we use in the later part of the paper. In section 3, we discuss the Penrose limit of the linear dilaton geometry and show that one can extract the high energy spectrum of the little string theory from the resulting plane wave geometry. Section 4 and 5 deal with understanding the basic properties of renormalization group flow in little string theories. In section 4, we consider a null geodesic in the cigar geometry, which is dual to the little string theory at finite energy densities, while in section 5, we consider a more general solution stemming from an array of M5-branes in M-theory. Section 6 deals with similar exercise for non-commutative theories, concentrating in particular on the case of the D3-brane. Finally, we end with some conclusions.

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2. Penrose limits

As mentioned in the introduction, the Penrose limit is defined as the spacetime in the neighbourhood of a null geodesic in a specific scaling limit, which we will discuss in detail below. The limit was first proposed in [13] in the context of classical general relativity and was extended to supergravity solutions in [17]. Recent discussions including many examples of Penrose limits for D-brane geometries can be found in [18, 19, 20]. We shall in the next subsection review the general formulation of the Penrose limit, following [17].

2.1 The Penrose limit

Given any spacetime with a metric and a null geodesic in the spacetime, one can define the Penrose limit in the following fashion. We first choose our coordinate system so that it is well adapted to the null geodesic. This may be done by choosing a coordinate u which is the affine parameter along a congruence of null geodesics, a coordinate v to be the distance between neighbouring geodesics, and finally x^i to denote the remainder of the coordinates. These coordinates are chosen so as to satisfy $g_{uu} = 0, g_{ux^i} = 0$ and $g_{uv} = N$ so that the general form for the metric is

$$ds^2 = N\left(2\,du\,dv + dv\left(dv + \sum_i B_i(u,v,x^i)dx^i\right) + \sum_{i,j} C_{ij}(u,v,x^i)\,dx^i\,dx^j\right), \qquad (2.1)$$

where N is a dimensionless parameter introduced to keep track of the scaling limit of various quantities. Later on, when we discuss spacetime backgrounds of D-branes, N will be the number of D-branes, which we will choose large so as to have a well behaved semi-classical approximation. In the Penrose scaling limit [13] we define the following rescaling of coordinates:

$$(u, v, x^i) \longrightarrow \left(u, \frac{v}{N}, \frac{x^i}{\sqrt{N}}\right), \qquad (2.2)$$

and take $N \to \infty$. In this limit we see that the dv^2 and the $dv\,dx^i$ terms drop out and we are left with the plane-wave metric

$$ds^2 = 2\,du\,dv + \sum_{i,j} C_{ij}(u)\,dx^i\,dx^j\,. \qquad (2.3)$$

Note that the metric functions C_{ij} now become functions of the coordinate u alone. This form of the metric is called the Rosen form [21].

However, this metric is typically not geodesically complete: one can not cover the full spacetime without encountering coordinate singularities. As these are not physical singularities, however, one can write the metric in a more suitable form which does cover the full spacetime, as noticed by [22]. This is called the Brinkman [23], or harmonic, form, and is given by

$$ds^2 = 2\,du\,dv + \left(\sum_{i,j} A_{ij}(u)\,x^i\,x^j\right)du^2 + \sum_i dx^i\,dx^i\,. \qquad (2.4)$$

This form has the virtue of exhibiting the deviations from flat spacetime more clearly, and it turns out to be more amenable to doing light-cone quantization in string theory.

It is worthwhile to note that there is an equivalent method to define the plane-wave limit. Specifically, instead of adapting the coordinate u to a congruence of null geodesics, one simply considers it to be the affine parameter along a single geodesic, so that $g_{uu} = 0$ only along the geodesic, and $g_{ux^i} = 0$, $g_{uv} = N$ as before. In the scaling limit, dv^2 and the $dv\,dx^i$ terms drop out, but we are left with a du^2 term, as well as some non-flat $dx^i\,dx^i$ terms. The resulting plane-wave metric will partially be in Rosen and partially in Brinkman form. We will find this procedure calculationally more convenient in the rest of the discussion.

One can explicitly write down the coordinate transformations relating the Brinkman and Rosen forms for the metric, as done e.g. in [20]. We will present the relevant formulae for a simple metric ansatz with the metric functions $C_{ij}(u)$ appearing in (2.3) being diagonal. Let us write the metric in Rosen coordinates as

$$ds^2 = 2\,d\bar{u}\,d\bar{v} + G(\bar{u})\,d\bar{x}^2\,. \qquad (2.5)$$

Then the following change of variables

$$\bar{u} = u\,, \qquad \bar{x} = \frac{x}{\sqrt{G(u)}}\,, \qquad \text{and} \qquad \bar{v} = v + \frac{1}{4}\frac{\partial_u G(u)}{G(u)}x^2\,, \tag{2.6}$$

casts the metric in the form

$$ds^2 = 2\,du\,dv + dx^2 + \frac{1}{2}x^2\left(\frac{1}{2}\frac{(\partial_u G(u))^2}{G(u)^2} + \partial_u\left(\frac{\partial_u G(u)}{G(u)}\right)\right)du^2\,. \tag{2.7}$$

This gives the plane-wave metric in the desired Brinkman form.

2.2 Penrose limit for a metric ansatz

Let us now see how the Penrose scaling limit works for a general spacetime of the type we will consider in the remainder of the paper. For illustrative purposes, we will keep the notation simple while considering a general enough form of the metric, specifically

$$ds^2 = N\left[-f(r)dt^2 + h(r)dr^2 + g(r)\left(\cos^2\theta\, d\psi^2 + d\theta^2 + \sin^2\theta\, d\Omega_q^2\right)\right] + dy_p^2\,, \tag{2.8}$$

where f, h, and g are arbitrary smooth functions of r. This is a static $(p+q+4)$-dimensional metric, which is spherically symmetric in $q+4$ directions and flat in remaining p directions, expressed in a diagonal form. Typically, we will consider the metric as arising from a stack of N D-branes; we will ultimately be interested in the large-N limit. For further notational simplicity, we'll now drop the subscripts p and q.

Consider a null geodesic in the (t, r, ψ) plane, at a fixed point in the y space and around the equator of the transverse q-sphere, i.e. $y^i \equiv 0, \theta = 0$. That such a geodesic is admissible, is obvious from the spacetime symmetries $y^i \to -y^i$ and $\theta \to -\theta$. This geodesic is generated by the tangent vector $\dot{t}\partial_t + \dot{r}\partial_r + \dot{\psi}\partial_\psi$, the dot denoting derivative with respect to the affine parameter λ along the geodesic, $\dot{} \equiv d/d\lambda$.

Since ∂_t and ∂_ψ are Killing vectors as apparent from (2.8), when contracted with a tangent vector to a geodesic, they define constants of motion along that geodesic. In particular, $E \equiv f(r)\dot{t}$ and $J \equiv g(r)\cos^2\theta\dot{\psi}$ are conserved quantities, corresponding to the energy and angular momentum. For a null geodesic, $p^2 = 0$, which yields the following equation for $\dot{r}$ at $\theta = 0$:

$$\dot{r}^2 - \frac{1}{h(r)}\left[\frac{E^2}{f(r)} - \frac{J^2}{g(r)}\right] = 0\,. \tag{2.9}$$

The second term defines an effective potential $V(r)$ for a one-dimensional motion of a particle, which one can easily analyze. In particular, the only geodesic at constant $r = r_0$ is such that $V'(r_0) = 0$.

A priori, (2.9) describes a 2-parameter family of geodesics, determined by the constants of motion E and J. However, we can rescale these by redefining the affine parameter so that we have only one physical parameter, $\ell \equiv J/E$, describing the geodesic. This is always possible for null geodesics because the scale drops out. Furthermore, from (2.9) we also see that ℓ is bounded: for $h(r) > 0$, $\ell^2 \leq g(r)/f(r)$ for all reachable r. We will assume that $\ell > 0$, and return to the case of $\ell = 0$, i.e. purely radial geodesics, at the end of this section.

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Let us now rewrite the metric in more suitable coordinates, adapted to the null geodesic. The coordinate transformation from $(t, r, \psi) \to (u, v, x)$ is achieved by:[1]

$$
\begin{aligned}
dr &= \frac{1}{\sqrt{h(r)}}\sqrt{\frac{1}{f(r)} - \frac{\ell^2}{g(r)}}\, du\,, \\
dt &= \frac{1}{f}\, du - dv + \ell\, dx\,, \\
d\psi &= \frac{\ell}{g} du + dx\,.
\end{aligned}
\tag{2.10}
$$

We can use the first equation to solve for

$$
u(r) = \int \frac{\sqrt{h(r)}}{\sqrt{\frac{1}{f(r)} - \frac{\ell^2}{g(r)}}} dr
\tag{2.11}
$$

and invert to obtain $r = r(u)$. The remaining functions f and g appearing in the second two equations of (2.10) are then to be viewed as functions of u.

Substituting this back into the metric (2.8), we obtain

$$
\begin{aligned}
ds^2 &= N\Bigg(- f\left[\frac{1}{f}\, du - dv + \ell\, dx\right]^2 + \left(\frac{1}{f} - \frac{\ell^2}{g}\right) du^2 + \\
&\quad + g\left\{\cos^2\theta\left[\frac{\ell}{g}\, du + dx\right]^2 + d\theta^2 + \sin^2\theta\, d\Omega^2\right\}\Bigg) + dy^2\,, \\
&= N\Bigg[2\, du\, dv - \frac{\ell^2}{g}\sin^2\theta\, du^2 + \left(-\ell^2 f + g - g\,\sin^2\theta\right) dx^2 - 2\ell\,\sin^2\theta\, du\, dx + \\
&\quad + \cdots dv^2 + \cdots dv\, dx + g\left(d\theta^2 + \sin^2\theta\, d\Omega^2\right)\Bigg] + dy^2\,,
\end{aligned}
\tag{2.12}
$$

where the $\cdots$ in the above will drop out momentarily. We now rescale the coordinates as suggested by (2.2):

$$
\begin{aligned}
u &\longrightarrow u\,, \qquad v \longrightarrow \frac{v}{N}\,, \qquad \theta \longrightarrow \frac{z}{\sqrt{N}}\,, \\
x &\longrightarrow \frac{x}{\sqrt{N}}\,, \qquad \Omega \longrightarrow \Omega\,, \qquad y \longrightarrow y\,.
\end{aligned}
\tag{2.13}
$$

Finally, taking the Penrose limit $N \to \infty$, yields the plane-wave metric

$$
ds^2 = 2\, du\, dv - \frac{\ell^2}{g} z^2\, du^2 + \left(-\ell^2 f + g\right) dx^2 + g\left(dz^2 + z^2\, d\Omega^2\right) + dy^2\,.
\tag{2.14}
$$

To simplify notation further, we rewrite the metric in the space transverse to the equator

[1] Here we are choosing *outgoing* null geodesics, corresponding to $\dot{r} > 0$; however, we could have just as well picked the incoming ones. Later, when we study a black hole geometry, this corresponds to geodesics emerging from the white hole, which is the equivalent of the geodesics falling into the black hole.

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on the original q-sphere as $dz^2 + z^2 d\Omega^2 = d\vec{z}^2$, so that $z^2 = \vec{z}^2$. The plane-wave metric then takes the simple form

$$ds^2 = 2\,du\,dv - \frac{\ell^2}{g}\,\vec{z}^2\,du^2 + \left(-\ell^2 f + g\right) dx^2 + g\,d\vec{z}^2 + dy^2\,. \tag{2.15}$$

The metric as written contains functions $f(r(u))$ and $g(r(u))$ implicitly as functions of the coordinate u. The explicit form of these may be quite complicated, and may not even have an analytic expression. Also note that, written in this implicit form, the above expression is independent of the function h. However, since the relation between r and u as determined by (2.11) depends on h, this is illusory.

To cast the metric in the Brinkman form we can follow the procedure outlined in the previous subsection. In particular, by applying the coordinate transformation suggested by (2.5) and (2.6) iteratively, the final metric (2.15) becomes

$$ds^2 = 2\,du\,dv + \left(\frac{1}{2}F_z\,\vec{z}^2 + \frac{1}{2}F_x\,x^2\right) du^2 + dx^2 + d\vec{z}^2 + dy^2 \tag{2.16}$$

with

$$\begin{aligned} F_z &= -2\frac{\ell^2}{g} + \left(\frac{g'}{g}\right)' + \frac{1}{2}\left(\frac{g'}{g}\right)^2\,, \\ F_x &= \left(\frac{(g-\ell^2 f)'}{g-\ell^2 f}\right)' + \frac{1}{2}\left(\frac{(g-\ell^2 f)'}{g-\ell^2 f}\right)^2\,, \end{aligned} \tag{2.17}$$

where the prime denotes differentiation with respect to u, $' \equiv d/du$.

Now let us briefly return to the special case of radial geodesic, $\ell = 0$. The geodesic eq. (2.9) now simplifies to $\dot{r}^2 = 1/f(r)h(r)$, and correspondingly the coordinate transformation $(t,r) \to (u,v)$ is given by

$$\begin{aligned} dr &= \frac{du}{\sqrt{f(r)h(r)}}\,, \\ dt &= \frac{1}{f}\,du - dv\,. \end{aligned} \tag{2.18}$$

The metric expressed in these coordinates is then simply

$$ds^2 = N\left[2\,du\,dv - f\,dv^2 + g\left(\cos^2\theta\,d\psi^2 + d\theta^2 + \sin^2\theta d\Omega^2\right)\right] + dy^2\,. \tag{2.19}$$

Rescaling the coordinates as in (2.13) and taking the large-N limit, we obtain the metric,

$$ds^2 = 2\,du\,dv + g(u)\left(dx^2 + d\vec{z}^2\right) + dy^2\,. \tag{2.20}$$

Note that this is automatically expressed in the Rosen form.

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3. Ultra high energies and little strings

Little string theory (LST) is one of the most important examples of non-local theories. This theory arises on the world-volume of NS5-branes when one considers a decoupling limit, $g_s \to 0$ with fixed α' [6, 7]. LSTs share many properties with usual string theory, such as T-duality and Hagedorn behaviour of density of states, but are nevertheless non-gravitational theories. As such, one wishes to uncover as many of their secrets as possible. Presently, we will see that we can use the techniques outlined above to learn something about the high energy spectrum of the theory.

To probe the high energy regime we will consider the holographic dual of little string theory, formulated in terms of string propagation in the linear dilaton background [8]. We will zoom into the high energy sector of the little string theory by considering a particular Penrose limit of the spacetime. Let us start by recalling some facts about the near-horizon geometry of NS5-branes.

The full NS5-brane metric in string frame and the dilaton are given by

$$\begin{aligned} ds^2_{\rm str} &= -dt^2 + dy_5^2 + A(r)\left(dr^2 + r^2 d\Omega_3^2\right) \,, \\ e^{2\Phi} &= g_s^2 A(r) \end{aligned} \tag{3.1}$$

with

$$A(r) = 1 + \frac{N l_s^2}{r^2}\,, \tag{3.2}$$

In addition the supergravity solution also has a NS-NS three form H, which is N times the volume form of the three sphere, where N is the number of NS5-branes.

The near-horizon limit ($r \to 0$) of the NS5-brane metric (3.1), is achieved by dropping the 1 in the harmonic function. This is the linear dilaton geometry,

$$ds^2_{\rm str} = N\, l_s^2 \left(-d\tilde{t}^{\,2} + \cos^2\theta\, d\psi^2 + d\theta^2 + \sin^2\theta\, d\phi^2 + \frac{dr^2}{r^2}\right) + dy_5^2\,, \tag{3.3}$$

where we have rescaled the time coordinate, $t = \sqrt{N} l_s\, \tilde{t}$, and written out the metric on the $\mathbf{S}^3$ as $d\Omega_3^2 = d\theta^2 + \cos^2\theta d\psi^2 + \sin^2\theta d\phi^2$, with $\theta = 0$ being the equator. Let us henceforth set $l_s \equiv 1$ and measure everything in the string units. Roughly speaking, the radial direction, which is the linear dilaton direction, holographically corresponds to the scale of the LST.

3.1 Penrose limit of linear dilaton geometry

To consider the appropriate plane-wave limit, we want to zero in onto a null geodesic in the above geometry (3.3). Since we first want to consider the LST at a fixed scale, we want our geodesic to sit at a constant value of the radial coordinate. Our null geodesic will therefore run along the equator of the three sphere transverse to the NS5-branes. This is the exact analog of the limit considered in [15] for $AdS_5 \times \mathbf{S}^5$, the near-horizon geometry of D3-branes. Using (2.9), we can check that there does exist a geodesic at constant value of the radial coordinate, provided the the angular momentum is saturated, namely $\ell = 1$.

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We now introduce new coordinates (u, v, x, z), making no changes to the coordinates ϕ and y_5,

$$\begin{aligned} \tilde{t} \pm \psi &= 2u, 2\frac{v}{N}, \\ r &= \sqrt{N} e^{x/\sqrt{N}}, \\ \theta &= \frac{z}{\sqrt{N}}, \end{aligned} \tag{3.4}$$

where, as explained above, we have chosen the coordinate u to be the affine parameter along the null geodesic, and the coordinate v to define a null, covariantly constant Killing vector in the plane wave limit. After taking the large-N limit, keeping the rescaled coordinates fixed, we obtain the metric

$$ds^2_{\rm str} = -4\, du\, dv - \vec{z}^{\,2} du^2 + d\vec{z}^{\,2} + dx^2 + dy_5^2\,, \tag{3.5}$$

where $d\vec{z}^{\,2} = dz^2 + z^2\, d\phi^2$. We note in passing that a trivial rescaling of v will cast the metric in (3.5) into the standard form of the plane-wave metric given in (2.16).

Also, in this limit, the dilaton becomes a constant,

$$e^{2\Phi} = g_s^2 e^{x/\sqrt{N}} \longrightarrow g_s^2\,, \tag{3.6}$$

and the non-trivial three-form NS-NS field scales to $H_{u12} = 3$, where by indices $(1, 2)$ we denote the coordinates in the $\vec{z}$ plane, which we will henceforth parametrize as (z^1, z^2). The fact that we have a constant H-field implies that $B_{12} = u$ on the world-sheet.

Finally, with a trivial rescaling, we can write the Penrose limit of the near-horizon geometry of the NS5-branes as

$$\begin{aligned} ds^2_{\rm str} &= -4\, du\, dv - \mu^2 \vec{z}^{\,2} du^2 + d\vec{z}^{\,2} + dx^2 + dy_5^2\,, \\ B_{12} &= \mu\, u\,. \end{aligned} \tag{3.7}$$

This defines an exact string background to all orders in world-sheet perturbation theory [24, 25] and was first derived by [26] and recently rederived in the context of Penrose limits of NS5-brane backgrounds by [27] and in [28].

3.2 Implications for little string theory

Thus far, we have considered a particular scaling of the metric corresponding to the near-horizon geometry of NS5-branes. To make contact with little string theory, we need to identify the precise meaning of the Penrose limit from the dual theory point of view.

We can relate the light-cone energy and momentum to the original energies and momenta as measured in the *string frame metric* (3.1). Recalling that $\partial_t = \frac{1}{\sqrt{N}}\partial_{\tilde{t}}$,

$$\begin{aligned} 2p^- &= -i\frac{\partial}{\partial u} = -i\left(\frac{\partial}{\partial \tilde{t}} + \frac{\partial}{\partial \psi}\right) = \tilde{E} - J = \sqrt{N}E - J\,, \\ 2p^+ &= -i\frac{\partial}{\partial v} = -\frac{i}{N}\left(\frac{\partial}{\partial \tilde{t}} - \frac{\partial}{\partial \psi}\right) = \frac{\tilde{E} + J}{N} = \frac{\sqrt{N}E + J}{N}\,. \end{aligned} \tag{3.8}$$

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If, in the original metric (3.1), we look at energies E of order $\sqrt{N}$ and angular momenta of order N, with the additional constraint that $\sqrt{N}E-J$ is finite, then we can describe all such states in little string theory by string propagation in the Nappi-Witten background (3.7). Thus, the claim is that the spectrum of strings in the background (3.7) is isomorphic to the little string spectrum in the regime $\sqrt{N}E \sim J \sim \mathbf{O}(N)$.

It is very simple to read off the spectrum of string theory in the background (3.7), [29]–[33]. The presence of the covariantly constant, null Killing vector allows us to quantize the theory in light-cone gauge [34]. We identify the world-sheet time τ with the light-cone coordinate u, $u = \tau$. In light-cone gauge, the Polyakov sigma model reduces to six free bosons and two bosons (z^1, z^2) which are "massive" and have a time dependent magnetic field. Let us concentrate on the latter, for the former fields are quantized in the usual fashion.

The light-cone Hamiltonian for the two bosons (z^1, z^2) is

$$\begin{aligned} H_{lc} = \frac{1}{4\pi\alpha'} \int d\tau \int_0^{2\pi\alpha' p^+} d\sigma & \left(\frac{1}{2} \left(\partial_\tau z^1\right)^2 - \frac{1}{2} \left(\partial_\sigma z^1\right)^2 - \frac{1}{2}\mu^2 (z^1)^2 + z^1 \leftrightarrow z^2 \right) + \\ & + i\mu \left(\tau \partial_\tau z^1 \partial_\sigma z^2 - z^1 \leftrightarrow z^2\right). \end{aligned} \tag{3.9}$$

By a change of variables these can be recast as free bosons with shifted oscillators. To see this, note that we can do away with the magnetic field term and the mass term by writing the lagrangian in terms of $Z = e^{i\mu\sigma}(z^1 + iz^2)$. The complex scalar field Z is then a free scalar on the world-sheet.

The shift is a result of the time dependent magnetic field on the world-sheet. The spectrum then is very similar to free string theory, modulo the shift, since at the end of the day we are left with free scalars.

The upshot is that the spectrum of little string theory in the very high energy regime is almost free string spectrum in flat space in ten dimensions. We are here considering energies of order $\sqrt{N} l_s$ in the sector with U(1) charge $J \sim N$. This was a hitherto unexplored regime of little string theory, and our result yields a prediction that little string theory ought to have states whose properties are identical to states of free string theory in 10 dimensions.

4. Penrose limits in the cigar

One of the problems with string propagation in the linear dilaton background is that as we move further down the throat, we encounter a local string coupling that is diverging, rendering the string perturbation analysis invalid. We actually got around the same in the previous section, by a judicious choice of null geodesic. By staying put at constant radial coordinate in the near-horizon metric (3.3), we managed to stay away from the strong coupling region. This involves choosing the constant radial coordinate to be large — precisely what we need if we are going to talk about the high energy spectrum of little string theory.

We now would also like to incorporate some effects of the non-trivial geometry in the radial direction arising due to the presence of the dilaton. However, to start with, we want to excise the strong coupling region. Later we shall also incorporate the strong coupling effects by studying the appropriate M-theory background, but for now, we want to stay

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within the realm of ten-dimensional supergravity for simplicity of the discussion. Such a geometry is immediately available to us in the form of two-dimensional black hole. This is the near-horizon limit of a non-extremal NS5-brane taken in an appropriate scaling limit.

The near-horizon geometry of the non-extremal NS5-brane is given by the metric

$$\begin{aligned} ds^2_{\rm str} &= -\left(1-\frac{r_0^2}{r^2}\right) dt^2 + dy_5^2 + \frac{Nl_s^2}{r^2}\left(\frac{dr^2}{1-\frac{r_0^2}{r^2}} + r^2 d\Omega_3^2\right), \\ e^{2\Phi} &= g_s^2 \frac{Nl_s^2}{r^2}\,. \end{aligned} \tag{4.1}$$

The decoupling limit [35] is defined as a double-scaling limit, such that the asymptotic value of the string coupling and the horizon radius are scaled to zero simultaneously, while keeping fixed the energy density above extremality in string units, i.e.

$$g_s \longrightarrow 0\,, \qquad r_0 \longrightarrow 0\,, \qquad \mu_{\rm lst} = \frac{r_0^2}{g_s^2\; l_s^2} = \text{fixed}\,. \tag{4.2}$$

To analyze this limit, it is convenient to rescale the time coordinate $t = \tilde{t}\sqrt{N} l_s$ and introduce a new radial coordinate $r = r_0 \cosh\sigma$, which in the scaling limit gives the near-horizon geometry

$$\begin{aligned} ds^2_{\rm str} &= Nl_s^2\left(-\tanh^2\sigma\; d\tilde{t}^2 + d\sigma^2 + d\Omega_3^2\right) + dy_5^2\,, \\ e^{2\Phi} &= \frac{N}{\mu_{\rm lst}\;\cosh^2\sigma}\,. \end{aligned} \tag{4.3}$$

Again, we can measure all quantities in the string units, so that $l_s \equiv 1$. The coordinate σ is adapted to the region outside the black hole horizon.

The geometry described by (4.3) is a convenient way to regulate the strong coupling region. One interpretation of this geometry is to work in the euclidean signature black hole, and then this is the holographic dual of little string theory at finite temperature; the temperature being held fixed at the Hagedorn value of $T_H = 1/2\pi\sqrt{N\alpha'}$. We could also work with the lorentzian cigar, which corresponds to little string theory at finite energy density.

4.1 Null geodesics, scalings and Penrose limit

We will now examine the Penrose limit of this geometry. First of all, let us write the geodesic equation. As before, we parameterize the 3-sphere by $d\Omega_3^2 = d\theta^2 + \cos^2\theta\, d\psi^2 + \sin^2\theta\, d\phi^2$ and consider a geodesic in the $(\tilde{t}, \sigma, \psi)$ plane, along the equator $\theta = 0$ of the 3-sphere. Proceeding in the usual way explained in section 2, we arrive at the geodesic equation,

$$-\frac{1}{\tanh^2\sigma} + \left(\frac{d\sigma}{du}\right)^2 + \ell^2 = 0\,. \tag{4.4}$$

A few points can be noted from this equation. If we insist on a geodesic at a constant value of σ, i.e. look for a massless particle orbiting around the transverse three sphere without radially moving in toward (or away from) the black hole, then the only such

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location is at $\sigma = \infty$. This corresponds to the asymptotically linear dilaton part of the geometry, where the presence of the black hole is irrelevant. This will lead to the geometry which we have studied in the previous section, the Nappi-Witten background.

Our aim will be to explore the modifications caused by the presence of the black hole. One interesting avenue which suggests itself for exploration is to look at null geodesics with a radial component in order to understand what happens to the little string theory under RG flow, since according to the usual holographic map, the radial motion ought to correspond to scale transformations. If we consider purely radial null geodesics, i.e. geodesics with no angular momentum on the three sphere, then using (4.4) and the results of section 2, it is easy to see that we end up with flat space with a linear dilaton proportional to u. This is an uninteresting geometry, as it encodes little information about the little string theory. Therefore we will consider geodesics which are radially infalling, as well as carrying some angular momentum along the 3-sphere. As we will see, the corresponding Penrose limit will lead to plane-wave geometries with a time dependent metric.

To proceed, we solve for the affine parameter as a function of the radial coordinate; a straightforward integration yields

$$\cosh(\sigma) = \frac{\ell}{\sqrt{1-\ell^2}} \sinh\left(\sqrt{1-\ell^2}\, u\right) . \tag{4.5}$$

Note that this implies that $\cosh(\sigma) = u$ in the limit $\ell = 1$. The limiting case $\ell \to 1$ is clearly a special case, since then the geodesic equation simplifies dramatically. Physically, this corresponds to the angular momentum being saturated.

Let us therefore first derive the explicit form of the Penrose limit metric for this special case, $\ell = 1$. The change of coordinates suggested by (2.10) is

$$\begin{aligned} \sigma &= \cosh^{-1} u\,, \\ \psi &= u + x\,, \\ \tilde{t} &= (u - \coth^{-1} u) - v + x\,. \end{aligned} \tag{4.6}$$

Substituting the above coordinate change and scaling the coordinates as in (2.13) followed by the large-N limit gives:

$$ds^2 = 2\, du\, dv - \vec{z}^{\,2}\, du^2 + \frac{dx^2}{u^2} + d\vec{z}^{\,2} + dy_5^2\,, \tag{4.7}$$

where we have used $\vec{z}$ to encode the (z, ϕ) plane, as in (2.15). For later convenience, we recast this into the Brinkman coordinates *via* the coordinate transformation (2.6),

$$x \longrightarrow ux\,, \qquad v \longrightarrow v - \frac{x^2}{2u}\,, \tag{4.8}$$

so that the resulting metric becomes

$$ds^2 = 2\, du\, dv + \left(\frac{2x^2}{u^2} - \vec{z}^{\,2}\right) du^2 + dx^2 + d\vec{z}^{\,2} + dy_5^2\,. \tag{4.9}$$

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Now let us go back and consider the more general case $0 \leq \ell \leq 1$. Let $\cosh(\sigma) = \ell/\sqrt{1-\ell^2}\sinh\left(\sqrt{1-\ell^2}u\right) \equiv g(u)$. The appropriate change of coordinates which generalizes (4.6) is given by integrating

$$\begin{aligned}
d\sigma &= \sqrt{\frac{g(u)^2}{g(u)^2-1} - \ell^2 du}\,,\\
d\psi &= \ell\, du + dx\,,\\
d\tilde{t} &= \frac{g(u)^2}{g(u)^2-1}du - dv + \ell dx\,.
\end{aligned} \tag{4.10}$$

Now, we go through the same procedure as above and obtain at the end of the day, the plane-wave metric in Brinkman coordinates as:

$$ds^2 = 2\,du\,dv + \left(2\frac{(1-\ell^2)}{\sinh^2(\sqrt{1-\ell^2}u)}x^2 - \ell^2\,\vec{z}^{\,2}\right)du^2 + dx^2 + d\vec{z}^{\,2} + dy_5^2\,, \tag{4.11}$$

Note that all of the above formulae reduce to the previous ones in the limit $\ell \to 1$. The $\ell = 0$ case is a singular limit; to keep the original $\cosh(\sigma)$ finite, $\sinh(\sqrt{1-\ell^2}\,u) \to \infty$. In this limit, the whole du^2 term vanishes, in agreement with our generic result (2.20).

The backgrounds (4.9) and (4.10) also have a non-trivial dilaton which is a function of the u coordinate alone, and a NS-NS B-field given as ,

$$\begin{aligned}
e^{2\Phi(u)} &= \frac{N}{\mu_{\rm lst}}\,\frac{1-\ell^2}{\ell^2\sinh^2(\sqrt{1-\ell^2}\,u)}\,,\\
B_{z_1 z_2} &= \ell\, u\,.
\end{aligned} \tag{4.12}$$

One can understand the geometry (4.10) as follows: in the Penrose limit we are zooming into the part of the spacetime in the neighbourhood of a massless particle trajectory. The coordinate u parametrizes the trajectory, and is basically the radial coordinate in the original cigar metric (4.3). The u dependence of the metric (4.10) is explained easily, for by construction we are considering geodesics moving along that direction. By virtue of zooming in on our geodesic around the equator of the $\mathbf{S}^3$, we have effectively decompactified it, and the $\mathbb{R}^2$ parametrized by $\vec{z} = (z^1, z^2)$. The longitudinal directions of the five-brane y_5 play no interesting role and simply go along for the ride. We note that the only difference of this metric from the Nappi-Witten background derived in the previous section is the term proportional to x^2 in g_{uu}. Importantly, in the limit $u \to \infty$ this vanishes, as it must from our discussion following the analysis of the geodesic eq. (4.4). We note that similar time-dependent plane-wave backgrounds were derived in [36].

4.2 String propagation

To study string propagation in (4.9) and (4.10) we will write the sigma model in light-cone variables. We may work with a more general metric of the form:

$$\begin{aligned}
ds^2 &= 2du\,dv + \left(f(u)\,x^2 - \ell^2\,\vec{z}^{\,2}\right)du^2 + dx^2 + d\vec{z}^{\,2} + dy_5^2\,,\\
\Phi(u) &= \phi(u)\,; \qquad 2\frac{\partial^2\phi(u)}{\partial u^2} = f(u)\,, \qquad B_{12} = \ell\, u\,.
\end{aligned} \tag{4.13}$$

String propagation in this geometry can be studied again in light-cone quantization with $u = \tau$. The sigma model is as discussed in the case of the extremal NS5-brane solution with the addition of the $f(u)x^2$ term arising from g_{uu}. If we write the sigma model in the light-cone, this leads to a time dependent mass term for the x field. The five longitudinal directions of the five-brane are massless scalars and the coordinates along the $\mathbb{R}^2$ spanned by (z^1, z^2) are free bosons modulo the shifted frequencies.

For the x-coordinate the world-sheet lagrangian reads:

$$\frac{1}{4\pi\alpha'}\int d\tau \int_0^{2\pi\alpha' p^+} d\sigma \left[\frac{1}{2}\left(\frac{\partial x}{\partial \tau}\right)^2 - \frac{1}{2}\left(\frac{\partial x}{\partial \sigma}\right)^2 - \frac{1}{2}f(\tau)x^2\right]. \tag{4.14}$$

The equation of motion for the x field on the world-sheet, with $x(\sigma,\tau) = e^{in\sigma}x_n(\tau)$ and $u = \tau$, is

$$\partial_\tau^2 x_n(\tau) + n^2 x_n(\tau) - f(\tau)\, x_n(\tau) = 0\,. \tag{4.15}$$

Although we have a non-trivial dilaton in the background, it does not explicitly enter the sigma model lagrangian, as the world-sheet curvature is zero in light-cone gauge. The dilaton affects the lagrangian by determining the function $f(\tau)$ in (4.14) as in (4.13). In (4.13), the dilaton is determined up to a term linear in u, which has no physical effect on string propagation.

The above lagrangian (4.14) describes string propagation in a time dependent background, which was to be expected since we are looking for geodesics that are moving out from the horizon in (4.3). String propagation in such backgrounds was studied in [34]. In such time-dependent backgrounds it was found that the string oscillators upon interacting with the plane wave can get excited. This is the benign form of particle production that one can associate with these backgrounds. One can calculate the probabilities for finding the string at an excited state at some time, given its initial state at an earlier time.

We will interpret this time dependent background as *real time RG* in the little string theory. The motivation is that in the holographic description of the little string theory, the σ direction in the background (4.3) is the direction of the RG flow in the dual theory. This is the analog of the statement that radial evolution in AdS is equivalent to RG flow in the boundary conformal field theory [37].

The claim is then the following. The solution of the sigma model with given u corresponds to studying the little string theory with a finite cut-off $\Lambda = u$. A simple consistency check is that in the limit of an infinite cut-off, $\Lambda = u \to \infty$, implying that the x field is becoming massless on the world-sheet. This is precisely what we expect from our study of high-energy of the little strings in the previous section.

One might be worried about the singularity at $u = 0$ in the metrics $\tilde{(}4.9)$ and (4.10). This singularity actually corresponds to the black hole singularity in the original spacetime (4.3) before taking the Penrose limit. One may see this explicitly by deriving the Penrose limit using the metric written in (4.1). As mentioned earlier, the cigar geometry corresponds to little string theory at finite energy density $\mu_{\rm lst}$. This means that there is an infra-red cut-off in the theory. From the spacetime point of view, we should

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therefore restrict ourselves to the region outside the black hole horizon, as the radial position of the horizon determines the scale of the infra-red cut-off. Thus, we should have $u \geq 1/\sqrt{1-\ell^2}\,\sinh^{-1}\left(\sqrt{1-\ell^2}/\ell\right)$. It may be worrisome that such an ansatz leads to considering a geodesically incomplete spacetime. However, even if u were allowed to go to 0, the spacetime would still be geodesically incomplete, since then we would encounter a real singularity.

So what are the questions that can be answered in the little string theory from analysing string propagation on (4.9) or (4.10)? We should quantize the theory (4.14) as usual, by solving the equation of motion (4.15) and imposing canonical commutation relations. One of the interesting questions we can ask is: Suppose we start in the vacuum state at $\Lambda = \infty$, what is the average occupation number of the string at any given cut-off? One simple approach is to realize that the wave equation for the x-field (4.15) can be mapped to a quantum mechanics problem in one-dimension wherein the energy is n^2 and the potential is $f(\tau)$. For small energies, i.e. small n, compared with the height of the potential at the infra-red cut-off, use a WKB approximation to determine the average occupation number as

$$\langle N_{\rm occ}\rangle = \exp\left(\int_{\Lambda}^{\Lambda_{\rm UV}} \sqrt{f(\tau)}\, d\tau\right). \tag{4.16}$$

In fact, we can obtain an exact answer, without recourse to a WKB approximation, for the cases where $f(\tau)$ leads to a solvable quantum mechanical problem, such as $f(\tau) = \mathrm{sech}^2(\tau)$ and $f(\tau) = \tanh(\tau)$. The average occupation number is then determined in terms of the Bogoliubov coefficients [34].

For the metric (4.9) with $f(\tau) = 2/\tau^2$ we can evaluate (4.16) to give $\langle N_{\rm occ}\rangle = (\Lambda_{\rm UV}/\Lambda)^{\sqrt{2}}$. If we heuristically identify the average occupation number with the number of degrees of freedom of little string theory at the scale λ, we then conclude that there are more degrees of freedom in little string theory in the infra-red than in the ultra-violet. While this would be bizarre for a local field theory, there are indications that such a feature might indeed hold for non-local theories.

In fact, for little string theory it is known from analysis of string propagation in the cigar geometry [38], that in addition to defining operators which are inserted at the asymptotic linear dilaton end, we can also have localized states near the tip of the cigar. These bound states show up as poles in correlation function of the operators [39, 40]. If they are to be thought of as living close to the tip of the cigar, then we will not be able to see them in the ultra-violet, but by moving into the cigar we will begin to encounter these states. But, this is precisely how we probe the geometry with the plane-wave, and therefore it should not be surprising that we have more states as we reach the infra-red.

5. Interpolating between little strings and $(2,0)$ SCFT

In the previous section we considered a situation wherein we regulated the strong coupling region of the linear dilaton geometry by studying the two-dimensional black hole. However, our geodesic could still enter into the region behind the horizon, and the metric as written in (4.9) and (4.10) had a region of strong coupling (not to mention a singularity) in the

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region of small u. Our next aim is to better this and find a smooth metric. We will take into account the effects of the strong coupling by studying the appropriate configuration in M-theory for the case of IIA little string theory.

5.1 M5-brane array

Type-IIA NS5-branes have their origins in M-theory as M5-branes transverse to the M-theory circle. When we are looking at the near-horizon geometry of the type-IIA NS5-branes, we account for the strong coupling region of the resulting linear dilaton geometry by lifting the IIA solution to M-theory. So we need to write down the supergravity solution in 11 dimensions corresponding to an array of N M5-branes localized at a point on the 11^{th} circle.

The metric for this configuration is readily written down. The space transverse to the M5-branes is now $\mathbb{R}^4 \times \mathbf{S}^1$; we let w be the radial coordinate on the $\mathbb{R}^4$ and z_{11} be the direction along the M-theory circle. The metric then reads

$$\begin{aligned} ds^2 &= l_p^2 \Big(\tilde{H}^{-1/3} \left[-dt^2 + dy_5^2 \right] + \\ &\quad + \tilde{H}^{2/3} \Big[dz_{11}^2 + dw^2 + w^2 (\cos^2\theta \, d\psi^2 + d\theta^2 + \sin^2\theta \, d\phi^2) \Big] \Big) , \end{aligned} \tag{5.1}$$

where

$$\tilde{H} = \sum_{n=-\infty}^{\infty} \frac{N}{\left[w^2 + (z_{11} - \frac{n}{l_s^2})^2 \right]^{3/2}} . \tag{5.2}$$

We have already taken a near-horizon limit by dropping the additive constant in the definition of the harmonic function $\tilde{H}$.

The geometry described by the above metric is such that asymptotically we have a linear dilaton region, where the size of the 11^{th} circle is small and we can trust the ten-dimensional supergravity or perturbative string theory. As we come into the strong coupling region, the 11^{th} circle opens out and there is an intermediate region where there is a rather complicated geometry. As we further descend towards the location of the M5-branes we find a major simplification; the geometry is now simply the near-horizon geometry of a stack of M5-branes, which is simply $AdS_7 \times \mathbf{S}^4$.

To see this explicitly, let us introduce new coordinates ρ and χ where

$$\begin{aligned} w &= 4N\rho^2 \sin(\chi) , \\ z_{11} &= 4N\rho^2 \cos(\chi) . \end{aligned} \tag{5.3}$$

Under this change of coordinates the metric becomes

$$\begin{aligned} ds^2 &= 4N^{2/3} l_p^2 \Big[H^{-1/3} \left(-dt^2 + dy_5^2 \right) + \\ &\quad + H^{2/3} \left\{ \rho^2 d\rho^2 + \frac{\rho^4}{4} \left(d\chi^2 + \sin^2\chi \left(\cos^2\theta \, d\psi^2 + d\theta^2 + \sin^2\theta \, d\phi^2 \right) \right) \right\} \Big] , \\ H &= 4^3 N^2 \tilde{H} = \sum_{n=-\infty}^{\infty} \frac{1}{\left[\rho^4 - \frac{2n}{\tilde{l}_s^2} \rho^2 \cos\chi + \frac{n^2}{\tilde{l}_s^4} \right]^{3/2}} . \end{aligned} \tag{5.4}$$

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with a rescaling $\tilde{l}_s = 2\sqrt{N} l_s$. This change of coordinates, while not completely essential to deciphering the limiting nature of the metric (5.1), helps in simplifying the analysis and in considering the Penrose limits of this geometry.

In the metric (5.1) the Anti-de Sitter region arises when distances are small compared to the string length l_s. This means that we can simplify the harmonic function by collapsing the infinite sum to just the contribution from the $n = 0$ piece. Then in the new coordinates (5.3) we find $H(\rho) = 1/\rho^6$. Direct substitution of H into (5.4) yields the metric of $AdS_7 \times \mathbf{S}^4$,

$$ds^2 = 4N^{2/3} l_p^2 \left[\rho^2 \left(-dt^2 + dy_5^2 \right) + \frac{d\rho^2}{\rho^2} + \frac{1}{4} d\Omega_4^2 \right]. \tag{5.5}$$

The $\mathbf{S}^4$ is parameterized by χ and by the coordinates of the $\mathbf{S}^3$ appearing in (5.1).

In the opposite limit, to extract the linear dilaton region of the array solution (5.1), we need to consider radial distances which are larger than all other distance scales, $w \gg l_p$. As this limit is easier to visualize in the original (w, z_{11}) coordinates, let us first consider what happens to the limiting metric in these before passing to the (ρ, χ) coordinates.

In the original geometry (5.1), we want to look at the region far away from the M5-branes, which means taking w large. From far away, the branes look uniformly smeared over the 11^{th} direction; in other words the sum in (5.2) becomes an integral, i.e. $\tilde{H} = 2N/w^2$. The metric (5.1) is then given by

$$ds^2 = l_p^2 (2N)^{-1/3} w^{2/3} \left(-dt^2 + dy_5^2 + \frac{2N}{w^2} \left[dz_{11}^2 + dw^2 \right] + 2N \, d\Omega_3^2 \right). \tag{5.6}$$

We see that as $w \to \infty$, the dz_{11}^2 term drops out, and after further rescaling, the final metric is conformally related to the NS5-brane metric (3.1).

In the new (ρ, χ) coordinates, the same limit means taking ρ large, but also $\tan\chi \gg 1$, so that $\chi \sim \pi/2$. In this limit, $H \sim 2/\rho^4$, so that the metric becomes

$$ds^2 = 2 l_p^2 (2N)^{2/3} \, \rho^{4/3} \left(-dt^2 + dy_5^2 + 2\frac{d\rho^2}{\rho^2} + \frac{1}{2} \left[d\chi^2 + \sin^2\chi \, d\Omega_3^2 \right] \right). \tag{5.7}$$

For $\chi \sim \pi/2$, the $\mathbf{S}^4$ part (in the square brackets) reduces to $d\Omega_3^2$, and the whole metric scales to the NS5-brane metric, as before. A point of some importance is that we naturally obtain the metric in the Einstein frame.

5.2 Penrose limits and M5-array solution

Our next step is to write down a plane-wave geometry which interpolates between the UV and the IR regions of the type-IIA little string theory. For this we will make use of the supergravity solution of an array of M5-branes in the form written in (5.4). The essential ingredient is the identification of the right null geodesic. This is a null geodesic which is in the (t, ρ, ψ) plane and is situated at $\chi = \pi/2$, $\theta = 0$, and $y_5 = 0$. The tangent vector to the geodesic is given by $\dot{t}\partial_t + \dot{\rho}\,\partial_\rho + \dot{\psi}\partial_\psi$. The existence of such a geodesic, despite the complicated dependence of the metric on the coordinate χ can be understood from the symmetry of the solution around $\chi = \pi/2$.

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Using the constants of motion, $E = H^{-1/3}\,\dot{t}$ and $J = H^{2/3}\rho^4/4\,\dot{\psi}$ at $\chi = \pi/2$ and $\theta = 0$, writing $\ell = J/E$ and rescaling $E = 1$, the geodesic equation, which may be obtained from (2.9), reads

$$\rho^2\dot{\rho}^2 = H^{-1/3}\left(1 - \frac{4\,\ell^2}{\rho^4 H}\right). \tag{5.8}$$

Adapting coordinates to this null geodesic following (2.10) then gives

$$\begin{aligned}
\rho\, d\rho &= H^{-1/6}\left(1 - \frac{4\ell^2}{\rho^4 H}\right)^{1/2} du\,, \\
d\psi &= \frac{4\ell}{\rho^4 H^{2/3}}\, du + dx\,, \\
dt &= H^{1/3}\, du - dv + \ell\, dx\,.
\end{aligned} \tag{5.9}$$

We can take the Penrose limit as discussed in section 2, to obtain the plane-wave metric in the Brinkman coordinates,

$$\begin{aligned}
ds^2 = {}& 2\,du\,dv + dx^2 + d\vec{z}^{\,2} + d\chi^2 + dy_5^2 + \\
&+\left(-\frac{4\ell^2}{\rho^4 H^{2/3}}\left(\vec{z}^{\,2} + \chi^2\right) + \frac{1}{2} f_{\vec{z}}(u)\,\vec{z}^{\,2} + \frac{1}{2} f_\chi(u)\chi^2 + \frac{1}{2} f_x(u) x^2 + \frac{1}{2} f_5(u)\, y_5^2\right) du^2\,,
\end{aligned} \tag{5.10}$$

where

$$\begin{aligned}
f_i(u) &= \frac{1}{2}\left(\frac{\partial_u h_i(u)}{h_i(u)}\right)^2 + \partial_u\left(\frac{\partial_u h_i(u)}{h_i(u)}\right), \\
h_5(u) &= H^{-1/3}(u)\,, \\
h_x(u) &= \left(H^{2/3}\frac{\rho^4}{4} - \ell^2 H^{-1/3}\right), \\
h_{\vec{z}}(u) &= h_\chi(u) = \frac{\rho^4}{4} H^{2/3}\,,
\end{aligned} \tag{5.11}$$

where we have written $H(u)$ as an implicit function of u. This may be determined by solving (5.8) for $\rho(u)$. We now demonstrate how to recover the limiting behaviour of the plane-wave metric (5.10). We will see that from this general form, we can isolate the plane-wave limit of the AdS region, as well as the linear dilaton region.

5.3 Relating to the AdS plane-wave

First we will recover the eleven-dimensional plane-wave associated with the AdS region. Recall that to obtain the AdS region from the metric (5.4) we are to look at length scales small compared to the string scale. Therefore we have $H(\rho) = 1/\rho^6$. Plugging this into the metric in (5.4), we recover the metric of $AdS_7 \times \mathbf{S}^4$ in Poincare coordinates as discussed earlier.

Given $H(\rho) = 1/\rho^6$, all we need to do is to solve for $\rho(u)$ using (5.8). We have

$$\rho\, d\rho = \rho\left(1 - 4\ell^2\rho^2\right) du \Longrightarrow \rho(u) = \frac{1}{2\ell}\sin(2\ell u)\,. \tag{5.12}$$

This implies that the functions $h_i(u)$, which determine the explicit form of the plane-wave

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metric as in (5.11), are given as

$$\begin{aligned} h_5(u) &= \rho^2(u) = \frac{1}{4\ell^2}\sin^2(2\ell u)\,, \\ h_x(u) &= \frac{1}{4}\cos^2(2\ell u)\,, \\ h_{\vec{z}}(u) &= h_\chi(u) = \frac{1}{4}\,. \end{aligned} \tag{5.13}$$

Therefore the resulting metric is:

$$ds^2 = 2\,du\,dv - \ell^2\left(4\,y_5^2 + 4\,x^2 + \vec{z}^{\,2} + \chi^2\right)du^2 + dy_5^2 + dx^2 + d\vec{z}^{\,2} + d\chi^2 \tag{5.14}$$

which we recognize as the usual plane-wave limit of $AdS_7 \times \mathbf{S}^4$, cf. [15].

5.4 Relating to the linear dilaton plane-wave

Let us now turn to the other limit of interest, namely the linear dilaton regime. This we recall is obtained by looking at the array of M5-branes from afar i.e. $\rho \gg 1$. Here we had $H \sim 2/\rho^4$, so we can again solve for $\rho(u)$ using (5.8), obtaining

$$\rho\,d\rho = \frac{\rho^{2/3}}{2^{1/6}}\left(1 - 2\ell^2\right)^{1/2} du\,. \tag{5.15}$$

Hence $\rho = (\frac{4}{3}2^{-1/6}\sqrt{1-2\ell^2}u)^{3/4}$, so that $H = a^3/u^3$, where $a = 3/2^{3/2}\sqrt{1-2\ell^2}$, and $4\ell^2/\rho^4 H^{2/3} = b/u$ with $b = 3\ell^2/2^{1/2}\sqrt{1-2\ell^2}$.

For the functions $h_i(u)$ which are crucial in determining the plane-wave metric as in (5.11), we have:

$$\begin{aligned} h_5(u) &= \frac{u}{a} \Longrightarrow f_5(u) = -\frac{1}{2u^2}\,, \\ h_x(u) &= u\,\ell^2\left(\frac{1}{b} - \frac{1}{a}\right) \Longrightarrow f_x(u) = -\frac{1}{2u^2}\,, \\ h_{\vec{z}}(u) &= h_\chi(u) = \frac{u\,\ell^2}{b} \Longrightarrow f_{\vec{z}}(u) = f_\chi(u) = -\frac{1}{2u^2}\,. \end{aligned} \tag{5.16}$$

With $\chi = 0, d\chi = 0$, this yields the metric

$$ds^2 = 2\,du\,dv + dx^2 + d\vec{z}^{\,2} + dy_5^2 - \frac{1}{4u^2}\left(\vec{z}^{\,2} + x^2 + y_5^2 + 4b\,u\,\vec{z}^{\,2}\right)du^2\,. \tag{5.17}$$

At first sight this does not appear to resemble the formulae for the plane-wave metric obtained in section 3. This is not much of a surprise, for the M-theory solution naturally descends into the Einstein frame metric for the type-IIA NS5-branes, which is related to the string frame metric appearing in (3.1) through a conformal transformation. In fact, the Einstein frame metric is given in (5.6) or (5.7). Although conformal transformations do not change the geodesic equation, they do affect the Penrose limit, since we are making use of conserved quantities to parametrize our solutions to the geodesic equation. They are affected because conformal transformations involve a local recalibration of our measuring devices. If we write down the Penrose limit of the geometry (5.7), for a null geodesic which carries angular momentum along the $\mathbf{S}^3$ and also at the same time is falling in into the linear dilaton region, we will indeed obtain the plane-wave metric (5.17).

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This demonstrates that the plane-wave geometry given in (5.11) is a plane wave geometry interpolating between the linear dilaton and the AdS plane waves, providing an explicit supergravity solution which can be used to study the properties of renormalization group flow in little string theory in the absence of an infra-red cut-off. Notice that unlike the case of the cigar geometry, in the infra-red region, there is now no singularity in the solution.

6. Non-commutative theories

Non-commutative field/string theories provide yet another example of non-local theories; one might naturally wonder whether considerations of Penrose limits of the corresponding holographic dual spacetime leads to some useful statements about these theories. We will try to study RG flows in these theories too, by considering geodesics with a radial component. Unfortunately, the resulting plane-wave geometry will prove to be quite messy and does not lend any insight into the physics of non-commutative theories.

As our final example, we therefore consider the Penrose limit of the supergravity dual to non-commutative Yang-Mills theories in four dimensions. We shall look at the near-horizon geometry of D3-branes with a constant NS-NS B-field on their world-volume. The supergravity solution for this geometry can be found for instance in [9] or in [10, c.f. eq. (2.7)] of the latter. We write the metric as:

$$ds^2 = R^2 \left[-r^2 dt^2 + \frac{dr^2}{r^2} + \cos^2\theta \, d\psi^2 + d\theta^2 + \sin^2\theta \, d\Omega_3^2 + r^2 \, dy_1^2 + \frac{r^2}{1 + a^4 r^4} dy_2^2 \right]. \qquad (6.1)$$

Note that this differs from the conventions of [10] for we have replaced their coordinates $u \to r$, $\tilde{x}_1 \to y_1$, $(d\tilde{x}_2^2 + d\tilde{x}_3^2) \to dy_2^2$, in addition to setting $\alpha' = 1$. In this case, $R^2 \sim \sqrt{N}$ is taken to be large.

We might wish to take a Penrose limit along a null geodesic in the (t, ψ) plane, staying at constant r, in analogy with the considerations in [15]. However, we will soon see that such a geodesic does not exist in this case. Therefore, let us consider the more general geodesic in the (t, r, ψ) plane, which moves both "radially" and along the equator of the $\mathbf{S}^5$, at $\theta = 0$, $x_i = 0$. The tangent vector to the geodesic is $\dot{t}\,\partial_t + \dot{r}\,\partial_r + \dot{\psi}\,\partial_\psi$, and gives the constants of motion $E \equiv r^2\,\dot{t}$ and $J \equiv \dot{\psi}$ at $\theta = 0$. We obtain the geodesic equation as in (2.9)

$$\dot{r}^2 = 1 - \ell^2 r^2 . \qquad (6.2)$$

Note that this is exactly the same geodesic equation as one would find for $AdS_5 \times \mathbf{S}^5$ in the Poincare coordinates, as it should be.

From the effective potential, it is now obvious that the geodesic can not stay at any constant $r > 0$. In fact, one can easily integrate the geodesic equation to find

$$r(u) = \frac{1}{\ell} \sin(\ell\, u) \qquad (6.3)$$

which in the limit $\ell \to 0$ correctly reduces to $r(u) = u$.

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Let us therefore consider this more general geodesic, and construct the corresponding plane-wave metric. The change of coordinates suggested by (2.10) is

$$\begin{aligned} r &= \frac{1}{\ell}\sin(\ell u)\,, \\ t &= -\ell\,\cot(\ell u) - v + \ell\, x\,, \\ \psi &= \ell u + x \end{aligned} \tag{6.4}$$

which, after substitution and rescaling as in (2.13) and further coordinate transform to cast the resulting plane-wave metric in Brinkman form, leads to the metric

$$\begin{aligned} ds^2 &= 2\,du\,dv - \ell^2\,\left(x^2 + \vec{z}^{\,2} + y_1^2 + g(u)\,y_2^2\right)du^2 + \\ &\quad + dx^2 + d\vec{z}^{\,2} + dy_1^2 + dy_2^2\,, \end{aligned} \tag{6.5}$$

where

$$g(u) \equiv \frac{\ell^8 - a^8\sin^8(\ell u) - 2a^4\sin^2(\ell u)\cos^2(\ell u)\left(a^4\sin^4(\ell u) - 5\ell^4\right)}{\left(\ell^4 + a^4\sin^4(\ell u)\right)^2}. \tag{6.6}$$

Several further comments are in order. First, setting $a = 0$, we recover the maximally supersymmetric plane-wave resulting from $AdS_5 \times \mathbf{S}^5$. Also we recover the same for small u, for in this limit, the distinction between the y_1 and the y_2 directions disappears. In both these cases, we are considering essentially the analog of the vanishing B-field. We will not write down the various form fields and the dilaton which are non-vanishing in the supergravity background (6.1) and simply note that they lead to non-vanishing and finite fields in the plane-wave background as guaranteed by the general properties of the Penrose limit [17].

Unfortunately, string theory in the background (6.6) remains intractable even in the light-cone gauge. This seriously hampers our initial objective of hoping to understand the features of RG flow in the non-commutative field theory using the Penrose limit.

7. Conclusion

In this paper we have studied various Penrose limits of string theory backgrounds that are dual to little string theory and non-commutative gauge theory. The resulting plane-wave metrics are, by construction, solutions to the low energy effective action of the type-II string theory, and thus provide consistent backgrounds for string propagation. By studying strings in these backgrounds one obtains information about specific sectors of the corresponding non-local theory. In particular, for little string theory we have identified a sector with high energy and R-charge that is dual to strings in the Nappi-Witten background, with a spectrum that is essentially that of a free string theory. It would of course be nice to verify this correspondence explicitly in a similar way as for $N = 4$ super Yang-Mills, but at present this is hard to do since very little is known about little string theory.

For little strings with a finite energy density we derived a one-parameter family of Penrose limits of the dual "cigar" geometry. An interesting feature of these plane-waves is the explicit dependence on the light-cone coordinate. As a result, the world-sheet theory

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of strings in these backgrounds is described in the light-cone gauge by scalars and fermions with mass terms that explicitly depend on the world-sheet time. From a dual perspective this time dependence is interpreted as a time-dependent RG-scale. String propagation in these backgrounds can be studied qualitatively using semi-classical methods. It would be interesting to perform a more quantitative analysis of these backgrounds and illuminate the dual RG-interpretation in more detail.

Another example, with an obvious dual RG-interpretation, is the plane-wave geometry that interpolates between the "static" plane-waves corresponding to little strings in the UV and the $(2,0)$ theory in the IR. The somewhat complicated form of this plane-wave metric, as well as the one corresponding to non-commutative gauge theory, makes a detailed analysis of the string theory very hard. We expect that the methods discussed in this paper, when applied to other backgrounds with dual interpretations, will yield plane-wave metrics with similar time-dependence. For example, one could consider the D1-D5 system and other intersecting brane configurations. It would be very interesting if among these one can find examples that leads to "solvable" (=integrable) time-dependent world-sheet theories.

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